Magnetic oscillations in graphene

Simon Becker (joint work with Maciej Zworski)

Univ. of Cambridge

Hexagonal quantum graph

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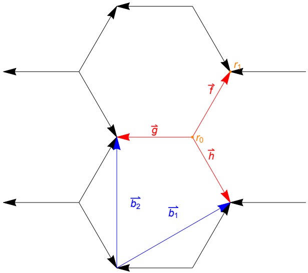

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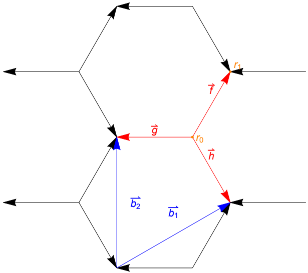

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Kuchment–Post '07

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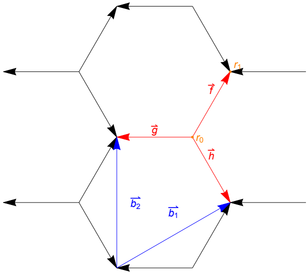

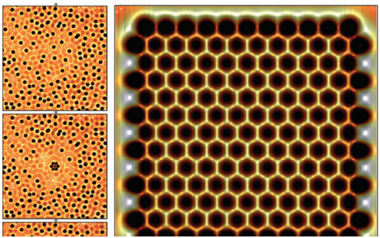

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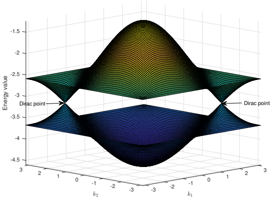
Kuchment–Post '07 Manoharan et al '12



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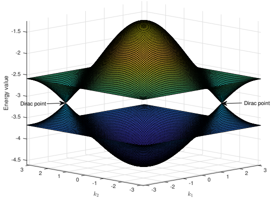

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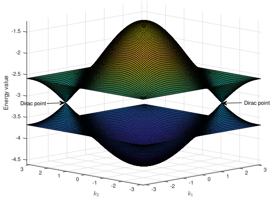

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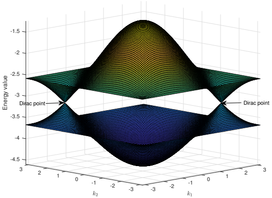

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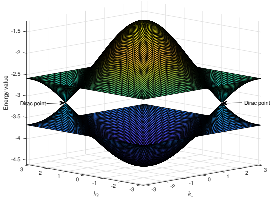

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Kuchment–Post '07

Fefferman–Weinstein '12, '14: 2D Schrödinger equation models

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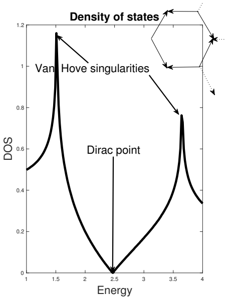

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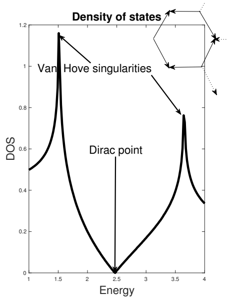

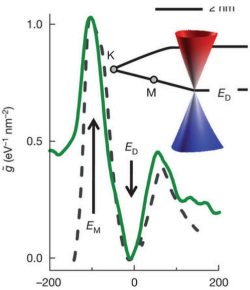

Quantum graph

Molecular graphene Manoharan et al '12

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Brüning–Geyler–Pankrashkin '07

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Remember also that the almost-Mathieu operator is

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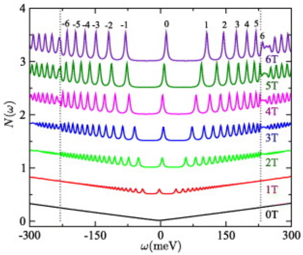

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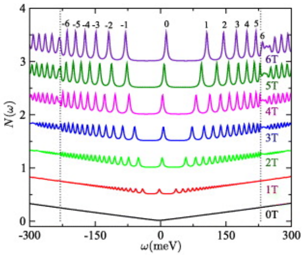
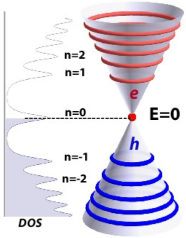


Pound et al '11,



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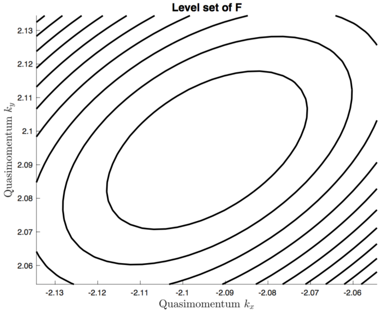

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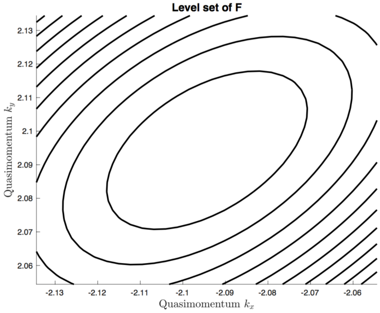

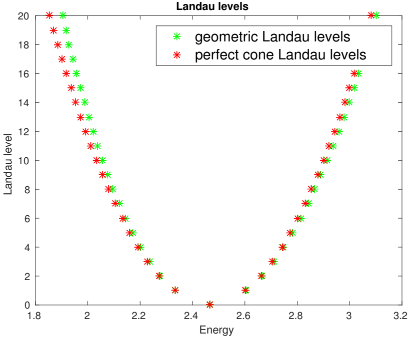

DOS is simple as $h \to 0$:

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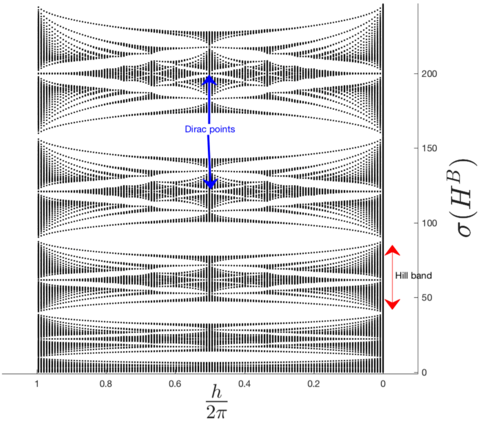

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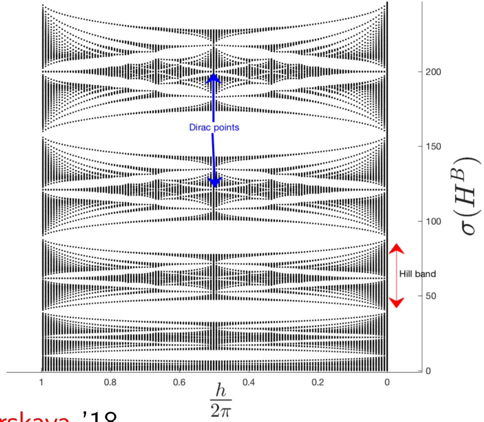

B.–Han–Jitomirskaya '18

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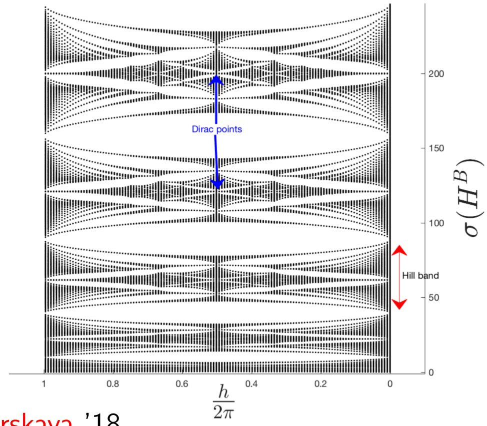

B.–Han–Jitomirskaya '18

Hofstadter '76 ... Avila–Jitomirskaya '09 ...

Magnetic (de Haas–van Alphen?) oscillations

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Differentiation can be justified for $\beta < h^{-M}$ (Helffer–Sjöstrand '90)

$$\int f(E)\rho_B(E)dE = \frac{h}{\pi\,|b_1 \wedge b_2|}\sum_{n\in\mathbb{Z}} f(E_n(h)) + \mathcal{O}_{\|f\|_{C^\alpha}}(h^\infty), \quad \alpha > 0$$

The proof follows the strategy of Helffer–Sjöstrand '90.

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Weinstein '77, Colin de Verdière '80, ... , Helffer–Robert '84

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Step 5 (the most technical). For $\lambda \in \mathrm{nbhd}_{\mathbb{C}}(\mathrm{I}) \setminus \mathbb{R}$,

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Compare to the formal expression:

$$\rho_B(E) = h\sum_{n\in\mathbb{Z}} (E - E_n(h) - i0)^{-1} - (E - E_n(h) + i0)^{-1}$$

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Comparison with numerics for the exact formula for rational h:

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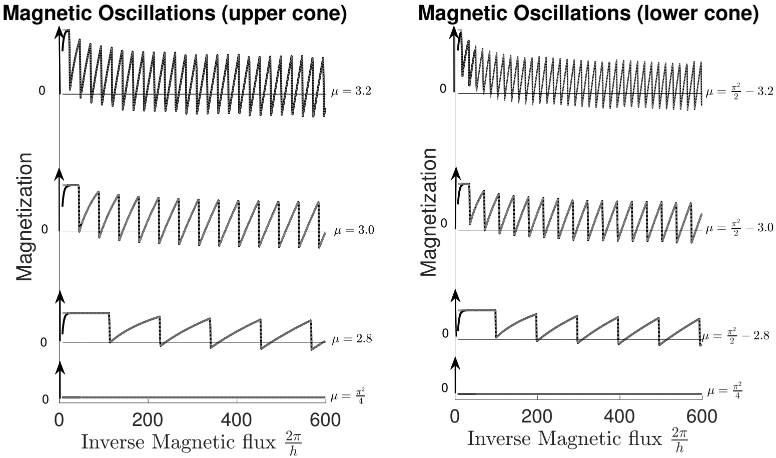

Thank you very much!

S.B. and Maciej Zworksi, (2018), Magnetic oscillations in a model of graphene, arXiv:1801.01931.

S.B., Rui Han, and Svetlana Jitomirskaya, (2018), Cantor spectrum of graphene in magnetic fields, arXiv:1803.00988.